Asymptotics for Provisioning Problems with a Large Number of Participants

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Abstract

We consider problems of provisioning a good to a number of participants who are able to communicate information about their private preferences for the good. The good may or may not be a public good. This provisioning is to be done in a manner that is incentive compatible, rational and feasible (in senses to be described). We show that as the number of participants becomes large there is a limiting problem, whose solution takes a simple form. For instance, it is near optimal to set a fixed fee and allow a participant to access the good if he is willing to pay this fee. The advantage of this is that the provisioning policy and fee structure can be easily communicated to the participants.

1 A problem of providing a public good

This paper is about problems of provisioning a good amongst a number of participants who are able to communicate information about their private preferences for the good. The good may or may not be a public good. This provisioning is to be done in a manner that is incentive compatible, rational and feasible (in senses that are described below). In Section 2 we show that as the number of participants becomes large there is a limiting problem, whose solution takes a simple form. For instance, it is near optimal to set a fixed fee and allow a participant to access the good only if he is willing to pay this fee. The advantage of this is that the provisioning policy and fee structure can be easily communicated to the participants. In Section 3 we describe some models to which our result can be applied. We prove the main theorem in Section 4 and discuss further directions in Section 5. Appendices A and B contain proofs for specialized and generalized versions of our main theorem.

We begin by summarising a model of Hellwig (2003). Consider n agents that bargain about the provision of a public good. A public good, once provided, can be enjoyed by all the agents at once. To provide the good in quantity (or, in some applications, quality) Q costs $c(n, Q)$. Once the good is provided, the net benefit to agent i is

$$\theta_i u(Q) - p_i$$

where p_i is the payment that the agent makes towards the cost of the good. The preference parameter θ_i is the realization of a random variable with prior distribution function F on

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[0, 1]. While F is known to all agents, the value of θ_i is known to agent i alone, We suppose that $\theta_1, \ldots, \theta_n$ are independent and identically distributed. Given $\theta = (\theta_1, \ldots, \theta_n)$, we ask what $Q(\theta)$ should be, and what payments, $p_1(\theta), \ldots, p_n(\theta)$, the agents should be required to make.

An allocation is said to be *feasible* if the sum of the payments always covers the cost, i.e.,

$$c(n, Q(\theta)) \leq \sum_{i=1}^{n} p_i(\theta) \tag{1}$$

for all $\theta \in [0, 1]^n$.

We say an allocation is *weakly feasible* if the expected sum of the payments covers the expected cost, i.e.,

$$\int c(n, Q(\theta))dF^n(\theta) \leq \sum_{i=1}^{n} \int p_i(\theta)dF^n(\theta)\,. \tag{2}$$

We suppose that the social planner may, if he wishes, exclude some agents from participating. Let $\pi_i(\theta)$ be the probability with which he chooses to include agent i given announced preferences θ. If exclusion is not an option for the planner, then we simply make the restriction $\pi_i(\theta) = 1$ for all θ, i.

We also consider the fact that there must be an incentive for each agent to participate. Agent i must expect to have a positive net benefit. This is the condition of *individual rationality*

$$\theta_i \int \pi_i(\theta_i, \theta_{-i})u(Q(\theta_i, \theta_{-i}))dF^{n-1}(\theta_{-i}) - \int \pi_i(\theta_i, \theta_{-i})p_i(\theta_i, \theta_{-i})dF^{n-1}(\theta_{-i}) \geq 0\,. \tag{3}$$

Here θ_{-i} is the vector of the preference parameters of the $n-1$ agents other than agent i.

Finally, we have an *incentive compatibility* condition, that agent i maximizes his expected net benefit by a truthful declaration of θ_i, i.e.,

$$\begin{aligned}
&\theta_i \int \pi_i(\theta_i, \theta_{-i})u(Q(\theta_i, \theta_{-i}))dF^{n-1}(\theta_{-i}) - \int \pi_i(\theta_i, \theta_{-i})p_i(\theta_i, \theta_{-i})dF^{n-1}(\theta_{-i}) \\
&\quad \geq \theta_i \int \pi_i(\theta_i, \theta_{-i})u(Q(\hat{\theta}_i, \theta_{-i})dF^{n-1}(\theta_{-i}) - \int \pi_i(\theta_i, \theta_{-i})p_i(\hat{\theta}_i, \theta_{-i})dF^{n-1}(\theta_{-i}) \geq 0
\end{aligned}$$

for all $\hat{\theta} \in [0, 1]$.

We can now define

$$V_i(\theta_i) = \int \pi_i(\theta_i, \theta_{-i})u(Q(\theta_i, \theta_{-i}))dF^{n-1}(\theta_{-i}) \tag{4}$$

and

$$P_i(\theta_i) = \int \pi_i(\theta_i, \theta_{-i})p_i(\theta_i, \theta_{-i})dF^{n-1}(\theta_{-i})\,. \tag{5}$$

The incentive compatibility condition implies two things. Firstly, we must have

$$[\hat{\theta}_i V_i(\hat{\theta}_i) - P_i(\hat{\theta}_i)] + [\bar{\theta}_i V_i(\bar{\theta}_i) - P_i(\bar{\theta}_i)] \geq [\hat{\theta}_i V_i(\bar{\theta}_i) - P_i(\bar{\theta}_i)] + [\bar{\theta}_i V_i(\hat{\theta}_i) - P_i(\hat{\theta}_i)]$$

If this were not true then it would be better to declare $\hat{\theta}_i$ when $\theta_i = \bar{\theta}_i$, and/or to declare $\bar{\theta}_i$ when $\theta_i = \hat{\theta}_i$. The above gives $(\hat{\theta}_i - \bar{\theta}_i)[V_i(\hat{\theta}_i) - V_i(\bar{\theta}_i)] \geq 0$ and hence we find the condition that (i) $V_i(\theta_i)$ is nondecreasing in θ_i.

Secondly, since θ_i maximizes $\theta_i V_i(\hat{\theta}_i) - P_i(\hat{\theta}_i)$ with respect to $\hat{\theta}_i$, we must have

$$\theta_i V_i'(\theta_i) - P_i'(\theta_i) = 0\,.$$

Integrating the above, we find a second condition, (ii):

$$P_i(\theta_i) = P_i(0) + \theta_i V_i(\theta_i) - \int_0^{\theta_i} V_i(\eta)\, d\eta\,. \tag{6}$$

Thus (i) and (ii) are necessary for incentive compatibility. It is easy to check that they are also sufficient.

Individual rationality is that $\theta_i V_i(\theta_i) - P_i(\theta_i) \geq 0$ for all θ_i. Considering this as $\theta_i \to 0$, we see that individual rationality requires $P_i(0) \leq 0$. Conversely, $P_i(0) \leq 0$ implies individual rationality via (6).

Now consider the problem of maximizing social welfare subject to the constraint that we use an allocation scheme which is weakly feasible[1], individually rational and incentive compatible. This means we are to maximize

$$\int \left[\sum_{i=1}^{n} \pi_i(\theta)\theta_i u(Q(\theta)) - c(n, Q(\theta)) \right] dF^n(\theta)\,. \tag{8}$$

Since the scheme is to be incentive compatible, we can deduce from (6) that the expected sum of the payments is given by

$$\begin{aligned}
\sum_{i=1}^{n} \int \pi_i(\theta_i, \theta_{-i}) p_i(\theta)\, dF^n(\theta) &= \sum_{i=1}^{n} \int P_i(\theta_i)\, dF(\theta_i) \\
&= \sum_{i=1}^{n} P_i(0) + \sum_{i=1}^{n} \int \left[\theta_i V_i(\theta_i) - \int_0^{\theta_i} V_i(\eta) d\eta \right] dF(\theta_i) \\
&= \sum_{i=1}^{n} P_i(0) + \sum_{i=1}^{n} \int \pi_i(\theta_i, \theta_{-i}) g(\theta_i) u(Q(\theta))\, dF^n(\theta) \qquad (9)
\end{aligned}$$

[1] In fact, without actually making the problem any more difficult, we can strengthen the constraint of weak feasibility to feasibility, since if $\{Q(\cdot), p_1(\cdot), \ldots, p_n(\theta)\}$ is incentive compatible, weakly feasible and individually rational then there is a new payment function $\hat{p}_i(\cdot)$ such that $\{Q(\cdot), \hat{p}_1(\cdot), \ldots, \hat{p}_n(\theta)\}$ is incentive compatible, feasible and individually rational. This is by an argument of Cramton et al. (1987), as follows. Explicitly, let

$$\begin{aligned}
\hat{p}_i(\theta) = P_i(\theta_i) + \frac{1}{n} c(Q(\theta)) - \frac{1}{n} \int c(Q(\theta_i, \theta_{-i}))\, dF^{n-1}(\hat{\theta}_{-i}) \\
-\frac{1}{n-1} \sum_{j \neq i} \left[P_j(\theta_j) - \int P_j(\hat{\theta}_j)\, dF(\hat{\theta}_j) \right] \\
+\frac{1}{n-1} \sum_{j \neq i} \frac{1}{n} \left[\int c(Q(\theta_j, \hat{\theta}_{-j}))\, dF^{n-1}(\hat{\theta}_{-j}) - \int c(Q(\hat{\theta}))\, dF^n(\hat{\theta}) \right].
\end{aligned} \tag{7}$$

Note that with this definition, $\sum_i \hat{p}_i(\theta_i) = c(Q(\theta)) + \sum_i \int P_i(\hat{\theta}_j)\, dF(\hat{\theta}_i) - \int c(Q(\hat{\theta}))\, dF^n(\hat{\theta}) \geq c(Q(\theta))$ so the new payment function is feasible. Also, $\hat{P}_i(\theta_i) = P_i(\theta_i)$, so the new payments are incentive compatible and individually rational.

where we define

$$g(\theta_i) = \theta_i - \frac{1 - F_i(\theta_i)}{f(\theta_i)}\,. \qquad (10)$$

Since the scheme is to be weakly feasible, we can use (9) to deduce that our problem is one of maximizing (8) subject to

$$-\sum_{i=1}^{n} P_i(0) \leq \sum_{i=1}^{n} \int \pi_i(\theta) g(\theta_i) u(Q(\theta))\, dF^n(\theta) - \int c(n, Q(\theta)\, dF^n(\theta) \qquad (11)$$

The maximization is with respect to a choice of the function $Q(\theta)$ and the constants $P_1(0)$, $\ldots, P_n(0)$. Restricting ourself to individually rational payments means we must take $P_i(0) \leq 0$ for all i. These enter only through their sum, which we may therefore take to be zero. Hence the problem reduces to one of maximizing (8) subject to the constraint

$$\int \left[\sum_{i=1}^{n} \pi_i(\theta) g(\theta_i) u(Q(\theta)) - c(n, Q(\theta)) \right] dF^n(\theta) \geq 0\,. \qquad (12)$$

We are to maximize (8) subject to (12) by pointwise choice of $Q(\cdot)$. From this we can calculate $V_i(\theta_i)$ and then the payments from (6) and (7). Provided $V_i(\theta_i)$ turns out to be nondecreasing we have then solved the problem of maximizing social welfare subject to use of a feasible, individually rational and incentive compatible scheme.

To solve the problem using Lagrangian methods, we must maximize the Lagrangian

$$\int \left[\sum_{i=1}^{n} \pi_i(\theta)(\theta_i + \lambda g(\theta_i)) u(Q(\theta)) - (1+\lambda) c(n, Q(\theta)) \right] dF^n(\theta) \qquad (13)$$

for some $\lambda > 0$. The maximization is carried out pointwise. That is, for each given θ, the values of $\pi_1(\theta), \ldots, p_n(\theta)$ and $Q(\theta)$ are chosen to maximize

$$A(\theta, \lambda) u(Q(\theta)) - c(n, Q(\theta)) \qquad (14)$$

where

$$A(\theta, \lambda) = \frac{\sum_{i=1}^{n} \pi_i(\theta)(\theta_i + \lambda g(\theta_i))}{1 + \lambda}\,. \qquad (15)$$

The fact that the coefficient $A(\theta, \lambda)$ should be maximized means that we should take $\pi_i(\theta) = 1$ if and only if $(\theta_i + \lambda g(\theta_i)) > 0$. Since $g(\theta_i)$ is nondecreasing, this means that agent i should be included if and only if θ_i exceeds some $\bar{\theta}(\lambda)$, where $\bar{\theta}(\lambda) + \lambda g(\bar{\theta}(\lambda)) = 0$. Note that $\bar{\theta}(\lambda)$ is increasing in λ, $A(\bar{\theta}(\lambda), \lambda)$ is decreasing in λ, and the $Q(\theta)$ which maximizes (14) is decreasing in λ.

2 An analysis for large n

The solution we have obtained is relatively simple, since we just admit participants whose preference parameters exceed some $\bar{\theta}(lambda)$. However, there is still the difficult matter of computing and communicating $Q(\theta)$ and the payments that the participants are to make. Fortunately, when n is large the problem becomes easier, because the payment is equal to a fixed fee, and this can be calculated and communicated before hearing the values $\theta_1, \ldots, \theta_n$. We now show why this is so. Recall that our problem is

$$\text{maximize} \int \left[\sum_{i=1}^{n} \pi_i(\theta)\theta_i u(Q(\theta)) - c(n, Q(\theta)) \right] dF^n(\theta) \tag{16}$$

subject to

$$\int \left[\sum_{i=1}^{n} \pi_i(\theta) g(\theta_i) u(Q(\theta)) - c(n, Q(\theta)) \right] dF^n(\theta) \geq 0\,. \tag{17}$$

To describe what happens when n is large we will apply the following theorem, taking ϕ and ψ such that when they are summed over i they produce the integrands of (16) and (17) respectively. That is, let

$$\phi(n, \theta_i, \pi_i(\theta), Q(\theta)) = \pi_i(\theta)\theta_i u(Q(\theta)) - c(n, Q(\theta))/n$$
$$\psi(n, \theta_i, \pi_i(\theta), Q(\theta)) = \pi_i(\theta) g(\theta_i) u(Q(\theta)) - c(n, Q(\theta))/n$$

Theorem 1 *Suppose $\theta_1, \ldots, \theta_n$ are independent and identically distributed random variables over $[0,1]$ with continuous distribution function F. Let $\theta = (\theta_1, \ldots, \theta_n)$. Let ϕ and ψ be any functions, $\phi, \psi : [0,1] \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$. Assume that $|c(Q)|$, $|\phi(\theta_i, Q_i, Q)|$ and $|\psi(\theta_i, Q_i, Q)|$ are bounded, say by B, for all (θ_i, Q_i, Q). Let us define*

$$\Phi_n = \max_{Q_1(\cdot), \ldots, Q_n(\cdot), Q(\cdot)} \int \left[\sum_{i=1}^{n} \phi(n, \theta_i, Q_i(\theta), Q(\theta)) \right] dF^n(\theta) \tag{18}$$

subject to

$$\int \left[\sum_{i=1}^{n} \psi(n, \theta_i, Q_i(\theta), Q(\theta)) \right] dF^n(\theta) \geq 0\,, \tag{19}$$

where the maximization is over the choice of $Q_1(\theta), \ldots, Q_n(\theta)$ and $Q(\theta)$ for each $\theta \in [0,1]^n$. Define

$$\hat{\Phi} = \max_{Q, Q_1(\cdot)} \int_0^1 \phi(n, \theta_1, Q_1(\theta_1), Q)\, dF(\theta_1) \tag{20}$$

subject to

$$\int_0^1 \psi(n, \theta_1, Q_1(\theta_1), Q)\, dF(\theta_1) \geq 0\,. \tag{21}$$

Then

$$n\hat{\Phi} \leq \Phi_n \leq n\hat{\Phi} + o(n)\,.$$

Moreover, note that If $Q^*, Q_1^*(\cdot)$ solves (20)–(21), then taking $Q(\theta) = Q^*$ for all θ and $Q_i(\cdot) = Q_1(\cdot)$ for all i solves (18)–(19) to within $o(n)$.

The proof of Theorem 1 is given in Section 5. For a proof of the theorem that is specialized to the model of Section 1, see Appendix A. We can find an approximate solution to the problem of Section 1 when n is large by solving a simple problem. We must choose $\bar{\theta}$ and Q to maximize

$$u(Q)\int_{\bar{\theta}}^{1}\theta_1 dF(\theta_1) - c(n,Q)/n$$

subject to

$$u(Q)\bar{\theta}(1-F(\bar{\theta})) - c(n,Q)/n \geq 0$$

3 Other problems

We now describe some other problems to which we can either apply Theorem 1 or generalize it to handle.

3.1 An extension

Suppose that the cost of providing the good depends not on n, but rather on the number of people included. Now the objective is

$$\int\left[\sum_{i=1}^{n}\pi_i(\theta)\theta_i u(Q(\theta)) - c\left(\textstyle\sum_j \pi_j(\theta), Q(\theta)\right)\right] dF^n(\theta) \qquad (22)$$

The analysis of this problem is very similar to that of the network model analyzed in Section 3.4. We must let Q be a vector, and permit contraints between Q and the Q_i. For example, in (22) we would take $Q_i(\theta) = \pi_i(\theta)$ and let Q be a vector of two components, one of which is the total quantity of the good provided (namely, $Q(\theta)$ in (22)) and the other is $\sum_j \pi_j(\theta)$.

3.2 File sharing

We consider a model in which agents have private preference parameters $\theta_1, \dots, \theta_n$ and the cost of operating a system in which Q files are shared is $c(Q)$. It is desired to maximize the social welfare function $\sum_i \theta_i u(Q) - c(Q)$.

As we have seen above, a near optimal policy when n is large is to admit only those agents whose preference parameter exceeds some $\bar{\theta}$. The problem is then

$$\underset{Q,\bar{\theta}}{\text{maximize}}\left\{u(Q)\int_{\bar{\theta}}^{1}(1-F(\theta))d\theta\right\}$$

subject to

$$n[1-F(\bar{\theta})]\bar{\theta}u(Q) - c(Q) \geq 0 \qquad (23)$$

So that Lagrangian methods can be applied to this problem, we rewrite the problem as

$$\underset{Q,\bar{\theta}}{\text{maximize}}\left\{\log u(Q) + \log\int_{\bar{\theta}}^{1}(1-F(\theta))d\theta\right\} \qquad (24)$$

subject to

$$\frac{c(Q)}{u(Q)} - n[1-F(\bar{\theta})]\bar{\theta} \leq 0 \qquad (25)$$

If each summand in (24) is concave and both $c(Q)/u(Q)$ and $-[1-F(\bar\theta)]\bar\theta$ in (25) are convex then the problem can be solved by Lagrangian methods. These facts are so for the particular case of $Q(c) = cQ^2$, $u(Q) = Q^{1/2}$ and $F(x) = x$ and we easily find that the optimum occurs where $\bar\theta = 1/8$ and $Q = (7n/64c)^{2/3}$.

3.3 WLAN peering model

Suppose that wireless LANs are built in k locations, and that there are potentially n_i LANS to be built in location i. The jth LAN in location i has a preference parameter θ_{ij} and the social welfare function is

$$\sum_{i,j} \pi_{ij}\theta_{ij} \sum_{\ell=1}^{k} u_\ell(Q_\ell)$$

where Q_ℓ is the quality of service provided in location ℓ. The idea is that the owner of a LAN in location i receives benefit when he roams in location j. The numbers $\{\theta_{ij}\}_{j=1}^{n_i}$ are taken to be iid samples from a distribution with distribution function F_i. The cost of providing quality Q_i in location i is $c(Q_i, \alpha_i \sum_{i,j} \pi_{ij})$, where the second argument expresses a dependence on the demand generated by customers roaming in location i. This cost must be covered by the payments made by LANS based in location i. Thus we have the problem of maximizing

$$\int_\Theta \sum_{i=1}^{k} \left[\sum_{j=1}^{n_i} \pi_{i,j}(\theta)\theta_{i,j} \textstyle\sum_{\ell=1}^{k} u_\ell(Q_\ell(\theta)) - c\left(Q_i(\theta), \alpha_i \sum_{i,j} \pi_{i,j}(\theta)\right) \right] \displaystyle\prod_i dF_i^{n_i}(\theta_{i\cdot})$$

subject to k constraints of the form

$$\int_\Theta \left[\sum_{j=1}^{n_i} \pi_{i,j}(\theta) g(\theta_{i,j}) \textstyle\sum_{\ell=1}^{k} u_\ell(Q_\ell(\theta)) - c\left(Q_i(\theta), \alpha_i \sum_{i,j} \pi_{i,j}(\theta)\right) \right] dF_i^{n_i}(\theta_{i\cdot}) \geq 0$$

The maximization is to be with respect to the functions $\{Q_i(\theta),\ i = 1,\dots,k\}$ and $\{\pi_{i,j}(\theta),\ i = 1,\dots,k,\ j = 1,\dots,k\}$.

The limiting problem (as n becomes large, with $(n_1,\dots,n_k) = (n\rho_1,\dots,n\rho_k)$ for some given $\rho_1,\dots,\rho_k$) is

$$\sum_{i=1}^{k} n_i \left[\textstyle\sum_{\ell=1}^{k} u_\ell(Q_\ell) \displaystyle\int_0^1 \pi_i(\theta_i)\theta_i dF_i(\theta_i) - \frac{1}{n_i} c\left(Q_i, \alpha_i \textstyle\sum_j n_j \int_0^1 \pi_j(\theta_j') dF_j(\theta_j')\right) \right]$$

This is to be maximized subject to k constraints of the form

$$\textstyle\sum_{\ell=1}^{k} u_\ell(Q_\ell) \displaystyle\int_0^1 \pi_i(\theta_i) g(\theta_i) dF_i(\theta_i) - \frac{1}{n_i} c\left(Q_i, \alpha_i \textstyle\sum_j n_j \int_0^1 \pi_j(\theta_j') dF_j(\theta_j')\right) \geq 0$$

with respect to $Q_1,\dots,Q_k$ and $\pi_1(\cdot),\dots,\pi_k(\cdot)$.

3.4 A network model

We might also formulate models in which the good cannot be shared. We continue to assume that the agents have preference parameters $\theta_1,\dots,\theta_n$. As a function of the declared values of these parameters, agents $1, \ldots, n$ are to be admitted or not and allocated exclusive quantities of the good, $Q_1(\theta), \ldots, Q_n(\theta)$, respectively, and the cost is a function of $\sum_j \pi_j(\theta)Q_j(\theta)$. The problem is now to

$$\text{maximize} \int \left[\sum_{i=1}^{n} \pi_i(\theta)\theta_i u(Q_i(\theta)) - c(n, \textstyle\sum_j \pi_j(\theta)Q_j(\theta)) \right] dF^n(\theta) \tag{26}$$

$$\tag{27}$$

subject to

$$\int \left[\sum_{i=1}^{n} \pi_i(\theta)g(\theta_i)u(Q_i(\theta)) - c(n, \textstyle\sum_j \pi_j(\theta)Q_j(\theta)) \right] dF^n(\theta) \geq 0 \tag{28}$$

$$\tag{29}$$

Further anaylsis of this model is continued in Appendix B.

4 Proof of Theorem 1

We now give the proof of the main result of this paper.
Proof. Suppose that the problem can be solved by maximizing a Lagrangian with Lagrange multiplier $\bar{\lambda}$. Then for $\bar{\lambda}$ and all other λ we have

$$\Phi_n = \max_{Q_1(\cdot),\ldots,Q_n(\cdot),Q(\cdot)} \int \left[\sum_{i=1}^{n} \phi(n, \theta_i, Q_i(\theta), Q(\theta)) + \bar{\lambda}\psi(n, \theta_i, Q_i(\theta), Q(\theta)) \right] dF^n(\theta) \tag{30}$$

$$\leq \max_{Q_1(\cdot),\ldots,Q_n(\cdot)Q(\cdot)} \int \left[\sum_{i=1}^{n} \phi(n, \theta_i, Q_i(\theta), Q(\theta)) + \lambda\psi(n, \theta_i, Q_i(\theta), Q(\theta)) \right] dF^n(\theta) \tag{31}$$

We will show that the integral in (31) is bounded above by $n\hat{\Phi}_n + o(n)$, where

$$\hat{\Phi}_n = \inf_{\lambda} \max_{Q_1(\cdot),Q} \int \left[\phi(n, \theta_1, Q_1(\theta_1), Q) + \lambda\psi(n, \theta_1, Q_1(\theta_1), Q) \right] dF(\theta_1) \tag{32}$$

We will need that $|\phi(n, \theta_1, Q_1, Q) + \lambda\psi(n, \theta_1, Q_1, Q)|$ to be bounded for all θ_1, Q_1 and Q.

By making the restriction that $Q_1 = \cdots = Q_n$, it is easy to see that $\Phi_n \geq n\hat{\Phi}_n$. Let us look at the proof of the bound $\Phi_n \leq n\hat{\Phi}_n + o(n)$. We prove the theorem when F is the uniform distribution. It is notationally more elaborate, but routine, to prove the theorem for general F.

Imagine dividing the interval $[0, 1]$ into k equal parts, defining

$$I_i = \left[\frac{i-1}{k}, \frac{i}{k} \right), \quad i = 1, \ldots, k.$$

Let the random variable X_i be the number of the $\theta_1, \ldots, \theta_n$ that are in I_i, Note that X_i has a binomial distribution with mean n/k, and that by Chebyshev's inequality we have

$$P(|X_i - n/k| > \epsilon) \leq \frac{n(1 - 1/k)(1/k)}{\epsilon^2}$$

We shall use this with $\epsilon = n^{2/3}$. Let us define the set $S = \{\theta : |X_i - n/k| \le n^{2/3},\ \text{for all } i\}$. Then

$$P(S^c) = P\left(\bigcup_{i=1}^{k}\left\{|X_i - n/k| > n^{2/3}\right\}\right) \le \sum_{i=1}^{k} P\left(\left\{|X_i - n/k| > n^{2/3}\right\}\right) \le \frac{1}{n^{1/3}}$$

Then we have for (42)

$$\max_{Q_1(\cdot),\ldots,Q_n(\cdot),Q(\cdot)} \int \left[\sum_{i=1}^{n} \phi(n,\theta_i,Q_i(\theta),Q(\theta)) + \lambda\psi(n,\theta_i,Q_i(\theta),Q(\theta))\right] dF^n(\theta) \tag{33}$$

$$\le \max_{Q_1(\cdot),\ldots,Q_n(\cdot),Q(\cdot)} \int_{\theta\in S} \left[\sum_{i=1}^{n} \phi(n,\theta_i,Q_i(\theta),Q(\theta)) + \lambda\psi(n,\theta_i,Q_i(\theta),Q(\theta))\right] dF^n(\theta) \tag{34}$$

$$+ \max_{Q_1(\cdot),\ldots,Q_n(\cdot),Q(\cdot)} \int_{\theta\notin S} \left[\sum_{i=1}^{n} \phi(n,\theta_i,Q_i(\theta),Q(\theta)) + \lambda\psi(n,\theta_i,Q_i(\theta),Q(\theta))\right] dF^n(\theta) \tag{35}$$

Since $P(S^c) \le 1/n^{2/3}$ we can bound (35) by $(1/n^{1/3})(nB)$, where B is bound on $|\phi(n,\theta_i,Q_i,Q) + \lambda\psi(n,\theta_i,Q_i,Q)|$. We bound (34) by

$$\max_{Q_1(\cdot),\ldots,Q_n(\cdot),Q(\cdot)} \int_{\theta\in S} \left[\sum_{i=1}^{n} \phi(n,\theta_i,Q_i(\theta),Q(\theta)) + \lambda\psi(n,\theta_i,Q_i(\theta),Q(\theta))\right] dF^n(\theta) \tag{36}$$

$$\le \max_{Q_1,\ldots,Q_n,Q,\theta\in S} \left[\sum_{i=1}^{n} \phi(n,\theta_i,Q_i,Q) + \lambda\psi(n,\theta_i,Q_i,Q)\right] \tag{37}$$

$$= \max_{\substack{Q_1,\ldots,Q_n,Q,\theta\in S,\\ \theta_1\in I_1,\ldots,\theta_k\in I_k}} \sum_{i=1}^{k} X_i[\phi(n,\theta_i,Q_i,Q) + \lambda\psi(n,\theta_i,Q_i,Q)] \tag{38}$$

$$\le \max_{\substack{Q_1,\ldots,Q_n,Q,\theta\in S,\\ \theta_1\in I_1,\ldots,\theta_k\in I_k}} (n/k)\sum_{i=1}^{k}[\phi(n,\theta_i,Q_i,Q) + \lambda\psi(n,\theta_i,Q_i,Q)] + \sum_{i=1}^{k}|X_i - (n/k)|B \tag{39}$$

$$\le \max_{\substack{Q_1,\ldots,Q_n,Q,\theta\in S\\ \theta_1\in I_1,\ldots,\theta_k\in I_k}} (n/k)\sum_{i=1}^{k}[\phi(n,\theta_i,Q_i,Q) + \lambda\psi(n,\theta_i,Q_i,Q)] + n^{2/3}kB\,. \tag{40}$$

Given any $\epsilon > 0$ we can choose k sufficiently large so that the intervals I_i have very small widths, of $1/k$, and so we can have (using continuity of f and approximation of an integral by a Reimamm sum)

$$\max_{\substack{Q_1,\ldots,Q_n,Q,\\ \theta_1\in I_1,\ldots,\theta_k\in I_k}} (1/k)\sum_{i=1}^{k}[\phi(n,\theta_i,Q_i,Q) + \lambda\psi(n,\theta_i,Q_i,Q)]$$

$$\le \max_{Q_1(\cdot),Q} \int_0^1 [\phi(n,\theta_1,Q_1(\cdot),Q) + \lambda\psi(n,\theta_1,Q_1(\cdot),Q)]\, dF(\theta_1) + \epsilon/2$$

Note that this requires that the term in square brackets be Riemann integrable. I imagine we will want Q restricted to an interval.

Given this k we can then choose n sufficiently large that $n^{2/3}B + n^{2/3}kB$ is less than $n\epsilon/2$. It follows, that given any $\epsilon > 0$ it is possible to choose k sufficiently large and then n sufficiently large to deduce that for n sufficiently large (but depending on λ),

$$\max_{Q_1(\cdot),\dots,Q_n(\cdot),Q(\cdot)} \int \left[\sum_{i=1}^{n} \phi(n,\theta_i,Q_i(\theta),Q(\theta)) + \lambda\psi(n,\theta_i,Q_i(\theta),Q(\theta)) \right] dF^n(\theta)$$
$$\leq n \max_{Q_1(\cdot),Q} \int_0^1 [\phi(n,\theta_1,Q_1(\cdot),Q) + \lambda\psi(n,\theta_1,Q_1(\cdot),Q)]\, dF(\theta_1) + n\epsilon$$

By taking an infimum over λ on the right hand side, assuming the infimum is achieved for some finite λ, we deduce $\Phi_n \leq n\hat{\Phi}_n + o(n)$. Note that if the infimum were achieved only as $\lambda \to \infty$ then we could not make this conclusion. ■

Remark. If we have the additional condition that $\phi(n,\theta_i,Q_i,Q)) = \phi^*(\theta_i,Q_i,Q))$ and $\psi(n,\theta_i,Q_i,Q)) = \psi^*(\theta_i,Q_i,Q))$ then we can replace $\phi(n,\theta_i,Q_i,Q))$ and $\phi(n,\theta_i,Q_i,Q))$ with $\phi^*(\theta_i,Q_i,Q))$ and $\phi^(\theta_i,Q_i,Q))$ respectively in (20) and (21).

5 On-going research

This is a working paper. There are further things upon which we intend to comment in the final version of this paper.

1. The asymptotic problem provides information about the best that can be obtained by schemes operating under the constraints of optimal incentive compatibility, waek feasibility and individually rationality. It allows us to see what is the benefit loss that occurs because of these constraints. The asymtotics tell us that the payments under an optimal scheme are tend to the value of an optimal fixed entrance fee. Compared to a fixed entrance fee scheme, an optimal scheme realizes very little extra benefit (typically only $O(\sqrt{n})$, or $O(1/\sqrt{n})$ per capita).

2. The constraints of weak feasibility and individual rationality are ones that must be met only on average. Might there be some nice formulation of our problem in which we have a repeated game and these constraints therefore seem more natural?

3. We have seen that argument of Cramton et al. (1987) demonstrates that weak feasibility implies feasibility. This does not appear to be provable when we take a model that allows exclusions. However, it is nearly optimal to runa fixed entrance fee scheme. By increasing the fee a bit we should be able to obtain $O(n)$ more revenue, which will cover any $O(\sqrt{n})$ uncovered cost and thereby ensure that we have feasibility with a very high probability.

4. There $\sqrt{n}$ results come from Hellwig's paper. Can we reproduce them more directly, along the lines of this paper? This would mean making more careful estimates and improving our essentially law of large numbers results to some sort of central limit theorem results.

5. There is an implicit assumption that the utility functions of the users is quasilinear in the payment. If the payment is euros, everything is euros. But what if we would like to make the payment in kind, say in channels or files? The same should hold, i.e., the utility should be quasilinear in files or channels. This rules out the case where the cost of individual users to provide f_i files is f_i^2 for instance. It works only if it is linear in f_i. Same for the WLANs. So we may not be in general able to balance cost with contributions in kind unless we have quasilinearity.

6. What happens with binary goods, when Q (or Q_i) must be 0 or 1?

7. Can we add dynamics to our model? What if participants arrive and depart? How might we reach a position with n participants? What if n is Poisson distributed?

8. There are other possible applications. We might consider network design problem with fixed and variable capacity, or delay problem (in which participants declare their value for delay and are charged appropriately).

References

Cramton, P., R. Gibbons, and P. Klemperer (1987). Dissolving a partnership efficiently. *Econometrica 55*, 615–632.

Hellwig, M. (2003, July). The impact of the number of participants on the provision of a public good. working paper, Department of Economics, University of Mannheim.

A Proof of Theorem 1 specialized to model of Section 1

Suppose that the optimal value is Φ_n and the problem can be solved by maximizing a Lagrangian with Lagrange multiplier $\bar{\lambda}$. Then for $\bar{\lambda}$ and all other λ we have

$$\Phi_n = \max_{\pi_1(\cdot),\ldots,\pi_n(\cdot),Q(\cdot)} \int \left[\sum_{i=1}^{n} \pi_i(\theta)(\theta_i + \bar{\lambda} g(\theta_i))u(Q(\theta)) - (1+\bar{\lambda})c(n,Q(\theta))\right] dF^n(\theta) \quad (41)$$

$$\leq \max_{\pi_1(\cdot),\ldots,\pi_n(\cdot),Q(\cdot)} \int \left[\sum_{i=1}^{n} \pi_i(\theta)(\theta_i + \lambda g(\theta_i))u(Q(\theta)) - (1+\lambda)c(n,Q(\theta))\right] dF^n(\theta) \quad (42)$$

We will show that the integral in (42) is bounded above by $\bar{\Phi}_n + o(n)$, where

$$\bar{\Phi}_n = n \cdot \inf_{\lambda} \max_{\pi_1(\cdot),Q} \int [\pi_1(\theta_1)(\theta_1 + \lambda g(\theta_1))u(Q) - (1+\lambda)c(n,Q)/n]\, dF(\theta_1) \quad (43)$$

We will need that $|\pi_1(\theta_1)(\theta_1 + \lambda g(\theta_1))u(Q)|$ is bounded for all θ_1 and Q. This condition will be satisfied, since $\pi_1(\theta_1) = 0$ when $\theta_1 + \lambda g(\theta_1) < 0$ (so it is here that we are relying on the possibility of exclusion) and also $\theta_1 + \lambda g(\theta_1)$ is bounded above (by the assumption that g is increasing and $g(1) = 1$). We will also need assumptions that $u(Q)$ and $c(n,Q)/n$ are bounded.

It is also easy to see that $\Phi_n \geq \bar{\Phi}_n$. Hence, we have $\bar{\Phi}_n \leq \Phi_n \leq \bar{\Phi}_n + o(n)$. Note that $\bar{\Phi}_n$ is n times the optimal value obtained when solving the following problem, *provided this problem can be solved by maximizing a Lagrangian.*

$$\underset{\pi_1(\cdot),Q}{\text{maximize}} \int [\pi_1(\theta_1)\theta_1 u(Q) - c(n,Q)/n]\, dF(\theta_1) \quad (44)$$

subject to

$$\int [\pi_1(\theta_1)g(\theta_1)u(Q) - c(n,Q)/n]\, dF(\theta_1) \geq 0 \quad (45)$$

Note that this is a simple problem. It is just to choose $\bar{\theta}$ and Q to maximize

$$u(Q)\int_{\bar{\theta}}^{1} \theta_1 dF(\theta_1) - c(n,Q)/n$$

subject to

$$u(Q)\bar{\theta}(1 - F(\bar{\theta})) - c(n,Q)/n \geq 0$$

Let us now look at the proof of the bound. We prove the theorem when F is the uniform distribution. It is notationally more elaborate, but routine, to prove the theorem for general F.

Imagine dividing the interval $[0,1]$ into k equal parts, defining

$$I_i = \left[\frac{i-1}{k}, \frac{i}{k}\right), \quad i = 1,\ldots,k.$$

Let the random variable X_i be the number of the $\theta_1,\ldots,\theta_n$ that are in I_i, Note that X_i has a binomial distribution with mean n/k, and that by Chebyshev's inequality we have

$$P(|X_i - n/k| > \epsilon) \leq \frac{n(1-1/k)(1/k)}{\epsilon^2}$$

We shall use this with $\epsilon = n^{2/3}$. Let us define the set $S = \{\theta : |X_i - n/k| \le n^{2/3}, \text{ for all } i\}$. Then

$$P(S^c) = P\left(\bigcup_i \left\{|X_i - n/k| > n^{2/3}\right\}\right) \le \sum_i P\left(\left\{|X_i - n/k| > n^{2/3}\right\}\right) \le \frac{1}{n^{1/3}}$$

Then we have for (42)

$$\max_{\pi_1(\cdot),\dots,\pi_n(\cdot),Q(\cdot)} \int \left[\sum_{i=1}^{n} \pi_i(\theta)(\theta_i + \lambda g(\theta_i))u(Q(\theta)) - (1+\lambda)c(n, Q(\theta))\right] dF^n(\theta) \tag{46}$$

$$\le \max_{\pi_1(\cdot),\dots,\pi_n(\cdot),Q(\cdot)} \int_{\theta\in S} \left[\sum_{i=1}^{n} \pi_i(\theta)(\theta_i + \lambda g(\theta_i))u(Q(\theta)) - (1+\lambda)c(n, Q(\theta))\right] dF^n(\theta) \tag{47}$$

$$+ \max_{\pi_1(\cdot),\dots,\pi_n(\cdot),Q(\cdot)} \int_{\theta\notin S} \left[\sum_{i=1}^{n} \pi_i(\theta)(\theta_i + \lambda g(\theta_i))u(Q(\theta)) - (1+\lambda)c(n, Q(\theta))\right] dF^n(\theta) \tag{48}$$

Since $P(S^c) \le 1/n^{2/3}$ we can bound (48) by $(1/n^{1/3})(nB)$ where B is an upper bound on $(\theta_i + \lambda g(\theta_i))u(Q)$. We bound (47) by

$$\max_{\pi_1(\cdot),\dots,\pi_n(\cdot),\,Q(\cdot)} \int_{\theta\in S} \left[\sum_{i=1}^{n} \pi_i(\theta)(\theta_i + \lambda g(\theta_i))u(Q(\theta)) - (1+\lambda)c(n, Q(\theta))\right] dF^n(\theta) \tag{49}$$

$$\le \max_{\pi_1,\dots,\pi_n,\,Q,\,\theta\in S} \left[\sum_{i=1}^{n} \pi_i(\theta_i + \lambda g(\theta_i))u(Q) - (1+\lambda)c(n, Q)\right] \tag{50}$$

$$= \max_{\substack{\pi_1,\dots,\pi_k,\,Q,\,\theta\in S,\\ \theta_1\in I_1,\dots,\theta_k\in I_k}} \sum_{i=1}^{k} X_i[\pi_i(\theta_i + \lambda g(\theta_i))u(Q)] - (1+\lambda)c(n, Q) \tag{51}$$

$$\le \max_{\substack{\pi_1,\dots,\pi_k,\,Q,\,\theta\in S,\\ \theta_1\in I_1,\dots,\theta_k\in I_k}} (n/k)\sum_{i=1}^{k} [\pi_i(\theta_i + \lambda g(\theta_i))u(Q)] + \sum_{i=1}^{k} |X_i - (n/k)|B' - (1+\lambda)c(n, Q) \tag{52}$$

$$\le \max_{\substack{\pi_1,\dots,\pi_k,\,Q,\\ \theta_1\in I_1,\dots,\theta_k\in I_k}} (n/k)\sum_{i=1}^{k} [\pi_i(\theta_i + \lambda g(\theta_i))u(Q) - (1+\lambda)c(n, Q)/n] + n^{2/3}kB' \tag{53}$$

where B' is a bound on $|\pi_i(\theta)(\theta_i + \lambda g(\theta_i))u(Q)|$).

Given any $\epsilon > 0$ we can choose k sufficiently large so that the intervals I_i have very small widths, of $1/k$, and so we can have (using continuity of f and approximation of an integral by a Reimamm sum)

$$\max_{\substack{\pi_1,\dots,\pi_n,\,Q,\\ \theta_1\in I_1,\dots,\theta_k\in I_k}} (1/k)\sum_{i=1}^{k} [\pi_i(\theta_i + \lambda g(\theta_i))u(Q) - (1+\lambda)c(n, Q)/n]$$

$$\le \max_{\pi_1(\cdot),Q} \int_0^1 [\pi_1(\theta)(\theta_1 + \lambda g(\theta_1))u(Q) - (1+\lambda)c(n, Q)/n]\, dF(\theta_1) + \epsilon/2$$

Note that this requires that the term in square brackets be Riemann integrable. I imagine we will want Q restricted to an interval.

Given this k we can then choose n sufficiently large that $n^{2/3}B + n^{2/3}kB'$ is less than $n\epsilon/2$. It follows, that given any $\epsilon > 0$ it is possible to choose k sufficiently large and then n sufficiently large to deduce that for n sufficiently large (but depending on λ),

$$\max_{\pi_1(\cdot),\dots,\pi_n(\cdot),\,Q(\cdot)} \int \left[\sum_{i=1}^{n} \pi_i(\theta)(\theta_i + \lambda g(\theta_i))u(Q(\theta)) - (1+\lambda)c(n, Q(\theta))\right] dF^n(\theta)$$
$$\leq n \max_{\pi_1(\cdot),\,Q} \int [\pi_i(\theta_1)(\theta_1 + \lambda g(\theta_1))u(Q) - (1+\lambda)c(n, Q)/n]\, dF(\theta_1) + n\epsilon$$

By taking an infimum over λ on the right hand side, assuming the infimum is achieved for some finite λ, we deduce $\Phi_n \leq \bar{\Phi}_n + o(n)$. Note that if the infimum were achieved only as $\lambda \to \infty$ then we could not make this conclusion.

B Proof of Theorem 1 generalized to model of Section 3.4

Suppose that the optimal value is Φ_n and the problem can be solved by maximizing a Lagrangian with Lagrange multiplier $\bar{\lambda}$. Then for $\bar{\lambda}$ and all other λ we have

$$\Phi_n = \max_{\substack{\pi_1(\cdot),\dots,\pi_n(\cdot),\\ Q_i(\cdot)}} \int \Bigg[\sum_{i=1}^{n} \pi_i(\theta)(\theta_i + \bar{\lambda} g(\theta_i))u(Q_i(\theta)) - (1+\bar{\lambda})c\left(n, \textstyle\sum_j \pi_j(\theta)Q_j(\theta)\right)\Bigg] dF^n(\theta) \tag{54}$$

$$\leq \max_{\substack{\pi_1(\cdot),\dots,\pi_n(\cdot),\\ Q_i(\cdot)}} \int \Bigg[\sum_{i=1}^{n} \pi_i(\theta)(\theta_i + \lambda g(\theta_i))u(Q_i(\theta)) - (1+\lambda)c\left(n, \textstyle\sum_j \pi_j(\theta)Q_j(\theta)\right)\Bigg] dF^n(\theta) \tag{55}$$

We will show that the integral in (42) is bounded above by $\bar{\Phi}_n + o(n)$, where

$$\bar{\Phi}_n = n \cdot \inf_{\lambda} \max_{\pi_1(\cdot),Q_1(\cdot)} \int \Bigg[\pi_1(\theta_1)(\theta_1 + \lambda g(\theta_1))u(Q_1(\theta_1)) - \frac{1}{n}(1+\lambda)c\left(n, n\int_0^1 \pi_1(\theta_1')Q_1(\theta_1')dF(\theta_1')\right)\Bigg] dF(\theta_1) \tag{56}$$

The proof follows the sames lines as before. We will need that $|\pi_1(\theta_1)(\theta_1+\lambda g(\theta_1))u(Q)|$ is bounded for all θ_1 and Q. This condition will be satisfied, since $\pi_1(\theta_1) = 0$ when $\theta_1+\lambda g(\theta_1) < 0$ (so it is here that we are relying on the possibility of exclusion) and also $\theta_1 + \lambda g(\theta_1)$ is bounded above (by the assumption that g is increasing and $g(1) = 1$). We will also need assumptions that $u(Q)$ and $c(n, Q)/n$ are bounded.

It is also easy to see that $\Phi_n \geq \bar{\Phi}_n$. Hence, we have $\bar{\Phi}_n \leq \Phi_n \leq \bar{\Phi}_n + o(n)$. Note that $\bar{\Phi}_n$ is n times the optimal value obtained when solving the following problem, *provided this problem can be solved by maximizing a Lagrangian.*

$$\underset{\pi_1(\cdot),Q}{\text{maximize}} \int \left[\pi_1(\theta_1)\theta_1 u(Q_1(\theta_1) - \frac{1}{n}c\left(n, n\int_0^1 \pi_1(\theta_1')Q_1(\theta_1')dF(\theta_1')\right)\right] dF(\theta_1) \tag{57}$$

subject to

$$\int \left[\pi_1(\theta_1)g(\theta_1)u(Q_1(\theta_1) - \frac{1}{n}c\left(n, n\int_0^1 \pi_1(\theta_1')Q_1(\theta_1')dF(\theta_1')\right)\right] dF(\theta_1) \geq 0 \tag{58}$$

Note that this is a simple problem. It is just to choose $\bar{\theta}$ and $Q_1(\cdot)$ to maximize

$$\int_{\bar{\theta}}^1 \theta_1 u(Q_1(\theta_1)dF(\theta_1) - \frac{1}{n}c\left(n, n\int_{\bar{\theta}}^1 Q_1(\theta_1')dF(\theta_1')\right) \tag{59}$$

subject to

$$\int_{\bar{\theta}}^{1} g(\theta_1)u(Q_1(\theta_1)dF(\theta_1) - \frac{1}{n}c\left(n, n\int_{\bar{\theta}}^{1} Q_1(\theta_1')dF(\theta_1')\right) \geq 0 \tag{60}$$

Let us now look at the proof of the bound. We prove the theorem when F is the uniform distribution. It is notationally more elaborate, but routine, to prove the theorem for general F.

Imagine dividing the interval $[0,1]$ into k equal parts, defining

$$I_i = \left[\frac{i-1}{k}, \frac{i}{k}\right), \quad i = 1, \ldots, k.$$

Let the random variable X_i be the number of the $\theta_1, \ldots, \theta_n$ that are in I_i, Note that X_i has a binomial distribution with mean n/k, and that by Chebyshev's inequality we have

$$P(|X_i - n/k| > \epsilon) \leq \frac{n(1-1/k)(1/k)}{\epsilon^2}$$

We shall use this with $\epsilon = n^{2/3}$. Let us define the set $S = \{\theta : |X_i - n/k| \leq n^{2/3},\text{ for all } i\}$. Then

$$P(S^c) = P\left(\bigcup_i \left\{|X_i - n/k| > n^{2/3}\right\}\right) \leq \sum_i P\left(\left\{|X_i - n/k| > n^{2/3}\right\}\right) \leq \frac{1}{n^{1/3}}$$

Then we have for (42)

$$\max_{\substack{\pi_1(\cdot),\ldots,\pi_n(\cdot),\\ Q_i(\cdot)}} \int \left[\sum_{i=1}^{n} \pi_i(\theta)(\theta_i + \lambda g(\theta_i))u(Q_i(\theta)) - (1+\lambda)c\left(n, \textstyle\sum_j \pi_j(\theta)Q_j(\theta)\right)\right] dF^n(\theta) \tag{61}$$

$$\leq \left(\max_{\substack{\pi_1(\cdot),\ldots,\pi_n(\cdot),\\ Q_i(\cdot)}} \int_{\theta\in S} + \max_{\substack{\pi_1(\cdot),\ldots,\pi_n(\cdot),\\ Q_i(\cdot)}} \int_{\theta\notin S}\right) \left[\sum_{i=1}^{n} \pi_i(\theta)(\theta_i + \lambda g(\theta_i))u(Q_i(\theta)) - (1+\lambda)c\left(n, \textstyle\sum_j \pi_j(\theta)Q_j(\theta)\right)\right] dF^n(\theta) \tag{62}$$

Since $P(S^c) \leq 1/n^{2/3}$ we can bound the second of the maximums in (62) by $(1/n^{1/3})(nB)$, where B is an upper bound on $(\theta_i + \lambda g(\theta_i))u(Q)$. We then bound the first maximum in (62) by

$$\max_{\substack{\pi_1,\ldots,\pi_n,\\ Q_1,\ldots,Q_n,\,\theta\in S}} \left[\sum_{i=1}^{n} \pi_i(\theta_i + \lambda g(\theta_i))u(Q_i) - (1+\lambda)c\left(n, \textstyle\sum_j \pi_j Q_j\right)\right] \tag{63}$$

$$= \max_{\substack{\pi_1,\ldots,\pi_k,\\ Q_1,\ldots,Q_k,\,\theta\in S}} \left[\sum_{i=1}^{k} X_i\pi_i(\theta_i + \lambda g(\theta_i))u(Q_i) - (1+\lambda)c\left(n, \textstyle\sum_{j=1}^{k} \pi_j X_j Q_j\right)\right] \tag{64}$$

$$\leq \max_{\substack{\pi_1,\ldots,\pi_k,\\ Q_1,\ldots,Q_k,\,\theta\in S}} \Bigg[\sum_{i=1}^{k} (n/k)\pi_i(\theta_i + \lambda g(\theta_i))u(Q_i) + |X_i - (n/k)|B'$$

$$- (1+\lambda)c\left(n, \textstyle\sum_{j=1}^{k} \pi_j (n/k) Q_j\right) + |X_j - (n/k)|B''\Bigg] \tag{65}$$

$$= n \cdot \max_{\substack{\pi_1,\ldots,\pi_k,\\ Q_1,\ldots,Q_k,\,\theta\in S}} \int \Bigg[\pi_1(\theta_1)(\theta_1 + \lambda g(\theta_1))u(Q_1(\theta_1)$$

$$- (1+\lambda)\frac{1}{n}c\left(n, n \textstyle\int_0^1 \pi_j(\theta_1')Q_1(\theta_1')dF(\theta_1')\right)\Bigg] dF(\theta_1) + o(n) \tag{66}$$

where B' and B'' are appropriate bounds and the last line follows by a Riemann sum approximation of the sort we have made previously. By taking an infimum over λ on the right hand side, assuming the infimum is achieved for some finite λ, we deduce $\Phi_n \leq \bar{\Phi}_n + o(n)$. Note that if the infimum were achieved only as $\lambda \to \infty$ then we could not make this conclusion.